Optimal split of orders across liquidity pools: a stochastic algorithm approach

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Outline

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Static modelling

The principle of a *Dark pool* is the following:

- It proposes a bid price with no guarantee of executed quantity at the occasion of an OTC transaction.
- Usually this price is lower than the bid price offered on the regular market.

So one can model the impact of the existence of N dark pools ($N \geq 2$) on a given transaction as follows:

- Let $V > 0$ be the random volume to be executed,
- Let $\theta_i \in (0,1)$ be the *discount factor* proposed by the dark pool i.
- Let r_i denote the percentage of V sent to the dark pool i for execution.
- Let $D_i \geq 0$ be the quantity of securities that can be delivered (or mase available) by the dark pool i at price $\theta_i S$.

Cost of the executed order

The rest of the order is to be executed on the regular market, at price S. Then the cost C of the whole executed order is given by

$$
\begin{aligned}
C &= S\sum_{i=1}^{N}\theta_i \min\left(r_i V, D_i\right) + S\left(V - \sum_{i=1}^{N}\min\left(r_i V, D_i\right)\right) \\
&= S\left(V - \sum_{i=1}^{N}\rho_i \min\left(r_i V, D_i\right)\right)
\end{aligned}
$$

where

$$\rho_i = 1 - \theta_i \in (0,1), i = 1, \ldots, N.$$

Mean Execution Cost

Minimizing the mean execution cost, *given the price* S, amounts to solving the following maximization problem

$$\max\left\{\sum_{i=1}^{N}\rho_i\mathbb{E}\left(S\min\left(r_iV, D_i\right)\right), r\in\mathcal{P}_N\right\} \tag{1}$$

where $\mathcal{P}_N := \left\{r=(r_i)_{1\le i\le N}\in\mathbb{R}_+^N \mid \sum_{i=1}^N r_i = 1\right\}$.

It is then convenient to *include the price* S *into both random variables* V *and* D_i by considering

$$\widetilde{V} := V\,S \quad \text{and} \quad \widetilde{D}_i := D_i S$$

instead of V and D_i.

The dynamical aspect

We consider the sequence $Y^n := (V^n, D_1^n, \ldots, D_N^n)_{n\geq 1}$.
We will take two types of stationarity assumptions on the sequence

$$(IID) \quad \equiv \quad \text{The sequence } (Y^n)_{n\geq 1} \text{ is i.i.d. with distribution } \nu = \mathcal{L}(V, D_1, \ldots, D_N) \text{ on } \left(\mathbb{R}_+^{N+1}, \mathcal{B}(\mathbb{R}_+^{N+1})\right).$$

$$(ERG)_i \quad \equiv \quad \begin{cases} (i) & \text{the sequence } (V^n, D_i^n)_{n\geq 1} \text{ is a stationary Feller homogeneous Markov chain with distribution } \mathcal{L}(V, D_i), \\ (ii) & \text{the sequence } (V^n, D_i^n)_{n\geq 1} \text{ is ergodic } i.e. \\ & \mathbb{P}\text{-}a.s. \quad \frac{1}{n}\sum_{k=1}^{n} \delta_{(V^k, D_i^k)} \stackrel{(\mathbb{R}_+^2)}{\Longrightarrow} \nu_i = \mathcal{L}(V, D_i), \end{cases}$$

Towards some solutions

There are two approaches to deal with this problem.

- A classical maximization (under constraints using a Lagrangian).
- An approach, somewhat more intuitive, based on a reinforcement principle; the algorithm is devised by R. Berenstein and C.-A. Lehalle in keeping with R. Almgren.

We will study both and try comparing their assets and drawbacks.

Outline

The mean execution function of a dark pool

Let $\varphi : [0,1] \to \mathbb{R}_+$ be the mean execution function of a single dark pool defined by

$$\forall r \in [0,1], \quad \varphi(r) = \rho\,\mathbb{E}\left(\min\left(rV, D\right)\right) \tag{2}$$

where $\rho > 0$, (V, D) is an $\mathbb{R}_+^2$-valued random vector defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. To ensure the consistency of the model, we assume that

$$V > 0 \ \mathbb{P} - a.s., \ \ V \in L^1(\mathbb{P}) \quad \text{and} \quad \mathbb{P}(D > 0) > 0 \tag{3}$$

The positivity of V means that we consider only true orders. The fact that D is not identically 0 means that the dark pool exists in practice.

The function φ is clearly concave, non-decreasing, bounded and if

$$\text{the distribution function of } \frac{D}{V} \text{ is continuous on } \mathbb{R}_+^*, \tag{4}$$

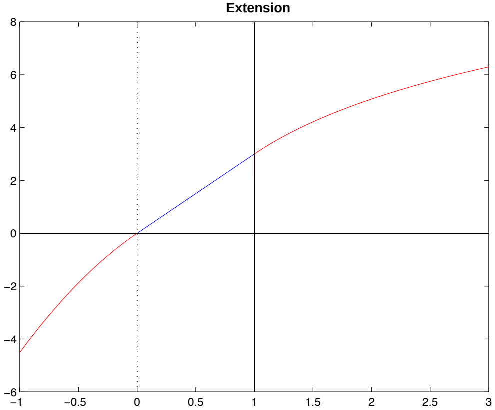
then φ is everywhere differentiable on the unit interval $[0,1]$ with

$$\varphi'(r) = \rho\mathbb{E}\left(\mathbb{1}_{\{rV<D\}}V\right), \;\; r \in [0,1]\,. \tag{5}$$

So the distribution of $\frac{D}{V}$ has no atom except possibly at 0. It can be interpreted as the fact the dark pool has no "quantized" answer to an order.
One extends φ on the whole real line into a concave non-decreasing function with $\lim_{\pm\infty} \varphi = \pm\infty$.

Extension of the mean execution function

One extends φ in the whole real line into a concave non decreasing function with $\lim_{\pm\infty} \varphi = \pm\infty$.



Optimal allocation of orders among N dark pools

Assume that V satisfies (3). We set for every $r = (r_1, \ldots, r_N) \in \mathcal{P}_N$,

$$\Phi(r_1, \ldots, r_N) := \sum_{i=1}^{N} \varphi_i(r_i).$$

where for every $i \in I_N = \{1, \ldots, N\}$,

$$\varphi_i(u) := \rho_i \mathbb{E}\left(\min\left(uV, D_i\right)\right), \quad u \in [0, 1]$$

Based on the extension of the functions φ_i, we can formally extend Φ on the whole affine hyperplan spanned by $\mathcal{P}_N$ *i.e.*

$$\mathcal{H}_N := \left\{ r = (r_1, \ldots, r_N) \in \mathbb{R}^N \mid \sum_{i=1}^{N} r_i = 1 \right\}$$

The Lagrangian Approach

We aim at solving the following maximization problem

$$\max_{r \in \mathcal{P}_N} \Phi(r) \tag{6}$$

The Lagrangian associated to the sole affine constraint is

$$L(r, \lambda) = \Phi(r) - \lambda \left(\sum_{i=1}^{N} r_i - 1 \right) \tag{7}$$

So, for every

$$i \in I_N, \quad \frac{\partial L}{\partial r_i} = \varphi_i'(r_i) - \lambda.$$

This suggests that any $r^* \in \arg\max_{\mathcal{P}_N} \Phi$ iff $\varphi_i'(r_i^*)$ is constant when i runs over I_N or equivalently if

$$\forall i \in I_N, \quad \varphi_i'(r_i^*) = \frac{1}{N} \sum_{j=1}^{N} \varphi_j'(r_j^*). \tag{8}$$

Existence of maximum

To ensure that the candidate provided by the Lagragian approach is the true one, we need an additional assumption on φ to take into account the bahaviour of Φ on the boundary of $\partial\mathcal{P}_N$.

Proposition 1

Assume that (V, D_i) satisfies (3) and (4) for every $i \in I_N$. Assume that the functions φ_i satisfy the following assumption

$$(\mathcal{C}) \equiv \min_{i\in I_N} \varphi_i'(0) > \max_{i\in I_N} \varphi_i'\left(\frac{1}{N-1}\right). \tag{9}$$

Then $\arg\max_{\mathcal{H}_N} \Phi = \arg\max_{\mathcal{P}_N} \Phi \subset int(\mathcal{P}_N)$ where

$$\arg\max_{\mathcal{P}_N} \Phi = \left\{r \in \mathcal{P}_N \mid \varphi_i'(r_i) = \varphi_1'(r_1), i = 1, \ldots, N\right\}.$$

Interpretation and Comments

Assumption $(\mathcal{C})$ is a kind of homogenity assumption on the rebates made by the involved dark pools. If we assume that

$$\mathbb{P}(D_i = 0) = 0 \quad \text{for every} \quad i \in I_N$$

(all dark pools buy or sell at least one security at the announced price!), then

$$(\mathcal{C}) \equiv \min_{i \in I_N} \rho_i > \max_{i \in I_N} \left(\rho_i \frac{\mathbb{E}\, V \mathbf{1}_{\{\frac{V}{N-1} \le D_i\}}}{\mathbb{E}\, V} \right)$$

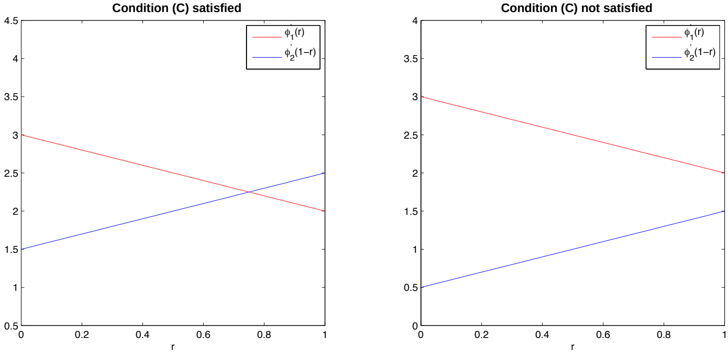
since $\varphi_i'(0) = \rho_i\, \mathbb{E}\, V$. In particular, it is always satisfied when

$$\rho_i = \rho, \;\; i \in I_N.$$

Interpretation of Condition ($\mathcal{C}$)

We consider the case where $N = 2$. We have then the two following derivatives

$$\varphi_1'(r_1) \quad \text{and} \quad \varphi_2'(r_2) = \varphi_2'(1 - r_1)$$



Design of the stochastic algorithm

Remark: $a_1, \ldots, a_N$ are equal iff $a_i = \frac{a_1+\cdots+a_N}{N}$, $\forall 1 \leq i \leq N$. Then using the representation of the derivatives φ_i' yields that, if Assumption $(\mathcal{C})$ is satisfied, then

$$r^* \in \arg\max_{\mathcal{P}_N} \Phi \Leftrightarrow$$

$$\forall i \in \{1, \ldots, N\}, \quad \mathbb{E}\left(V\left(\rho_i \mathbb{1}_{\{r_i^* V < D_i\}} - \frac{1}{N}\sum_{j=1}^N \rho_j \mathbb{1}_{\{r_j^* V < D_j\}}\right)\right) = 0.$$

Consequently, this leads to the following recursive zero search procedure

$$r_i^{n+1} = r_i^n + \gamma_{n+1} H_i(r^n, Y^{n+1}), \; r^0 \in \mathcal{P}_N, \; i \in I_N, \tag{10}$$

where for $i \in I_N$, every $r \in \mathcal{P}_N$, every $V > 0$ and every $D_1, \ldots, D_N \geq 0$,

$$H_i(r, Y) = V \left(\rho_i \mathbb{1}_{\{r_i V < D_i\}} - \frac{1}{N} \sum_{j=1}^{N} \rho_j \mathbb{1}_{\{r_j V < D_j\}} \right)$$

where $(Y^n)_{n \geq 1}$ is a sequence of random vectors with non negative components such that, for every $n \geq 1$ and $i \in I_N$, $(V^n, D_i^n) \stackrel{d}{=} (V, D_i)$.

The underlying idea of the algorithm

is to reward the dark pools which outperform the mean of the N dark pools by increasing the allocated volume sent at the next step (and conversely).

Constraint Problem

In this algorithm, we took into account the constraint

$$\sum_{i=1}^{N} r_i = 1,$$

but not

$$r_i > 0, \ \forall 1 \leq i \leq N.$$

So the algorithm may exit from the simplex $\mathcal{P}_N$ stable. To overcome this problem, we have two possibilities

1. Use a Lyapunov function and a strong *mean-reverting* assumption out of $\mathcal{P}_N$: this solution is simpler from a mathematical point of view.
2. Force the coefficients r_i to stay in $\mathcal{P}_N$ by a truncation-projection procedure: this solution is more efficient for users.

Convergence Theorem

Theorem 1

Assume that $V \in L^2(\mathbb{P})$ and that Assumption $(\mathcal{C})$ holds. Let $\gamma := (\gamma_n)_{n\geq 1}$ be a step sequence satisfying the usual decreasing step assumption

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty.$$

Let $(Y^n)_{n\geq 1}$ be an i.i.d. sequence defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then, there exists an $\arg\max_{\mathcal{P}_N} \Phi$-valued random variable r^∞ such that

$$r^n \xrightarrow{a.s.} r^\infty.$$

Rate of convergence (Central Limit Theorem)

To establish a CLT, we need to ensure the existence of the Hessian of the objective function Φ. This needs further assumption on a couple (V, D) which is that the conditional disctribution function

$$F_{_D}(u\,|\,V = v, D > 0) := \mathbb{P}(D \leq u\,|\,V = v, D > 0),\ u \geq 0,\ v > 0,$$

admits a density $f_{_D}(u, v)$ such that and satisfying

$$\left\{\begin{array}{ll} (i) & F_{_D}(u\,|\,V = v, D > 0) = \int_0^u f_{_D}(u', v)du' \\ (ii) & \text{for every } v > 0,\ u \mapsto f_{_D}(u, v) \text{ is continuous and positive,} \\ (iii) & \forall\,\varepsilon \in (0,1),\quad \sup_{\varepsilon V \leq u \leq V/\varepsilon} f_{_D}(u, V)V^2 \in L^1(\mathbb{P}). \end{array}\right. \tag{11}$$

Central Limit Theorem

Theorem 2

Assume that Assumption (11) holds for every $(V, D_i), i \in I_N$ and that $V \in L^{2+\delta}(\mathbb{P})$, $\delta > 0$. Set $\gamma_n = \frac{c}{n}, n \geq 1$ with $c > \frac{1}{2\Re(\lambda_{\min})}$, where $\lambda_{\min}$ denotes the eigenvalue of $A^\infty := -Dh(r^\infty)_{|1^\perp}$ with the lowest real part. Then

$$\sqrt{n}\,(r^n - r^\infty) \xrightarrow{\mathcal{L}} \mathcal{N}(0, c\Sigma_*) \tag{12}$$

where the asymptotic variance is given by

$$\Sigma_* = \int_0^\infty e^{u(A^\infty - \frac{Id}{2c})}\Sigma^\infty e^{(A^\infty - \frac{Id}{2c})^t} du$$

where $\Sigma^\infty = \mathbb{E}\left(H(r^\infty, V, D_1, \ldots, D_N)\, H(r^\infty, V, D_1, \ldots, D_N)^t\right)_{|1^\perp}$ and $\left(A^\infty - \frac{Id}{2c}\right)^t$ stands for the transpose operator of $A^\infty - \frac{Id}{2c} \in \mathcal{L}(1^\perp)$.

Convergence in (ERG) setting

We assume that for every $i \in \{1, \ldots, N\}$, the sequence $(V^n, D_i^n)_{n\geq 1}$ satisfies $(ERG)_i$ with a limiting distribution (V, D_i) satisfying the consistency assumption (3) and the continuity assumption (4), which implies by standard weak convergence arguments that for every $i \in \mathcal{I}_N$ and every $u \in \mathbb{R}_+$,

$$\frac{1}{n}\sum_{k=1}^{n} V^k \mathbf{1}_{\{uV^k < D_i^k\}} - \mathbb{E}(V\mathbf{1}_{\{uV \leq D_i\}})) \xrightarrow{a.s.} 0$$

since the (non-negative) function $f_u(v, y) := v\mathbf{1}_{\{uv \leq y\}}$ is $\mathbb{P}_{(V,D_i)}$-*a.s.* continuous and $O(v)$ as $v \to \infty$ by (4).

We assume that there exists an exponent $\alpha_i \in (0, 1)$ such that

$$\forall\, u \in \mathbb{R}_+, \quad \frac{1}{n}\sum_{k=1}^{n} V^k \mathbf{1}_{\{uV^k < D_i^k\}} - \mathbb{E}(V\mathbf{1}_{\{uV < D_i\}}) \overset{a.s.\ \&\ \text{in}\ L^2(\mathbb{P})}{=} O(n^{-\alpha_i}) \tag{13}$$

Convergence Theorem

Theorem 3

Let $(V_n, D_n^1, \ldots, D_n^N)_{n\geq 1}$ be a stationary Feller homogeneous Markov chain and assume that $\sup_{n\geq 0} \mathbb{E}(V^n)^4 < +\infty$. Furthermore, assume that for every $i \in \mathcal{I}_N$, the sequence $(V^n, D_i^n)_{n\geq 1}$ is ergodic at rate $\alpha_i \in (0,1)$ toward (V, D_i). Assume that the distribution of (V, D_i) satisfies the consistency assumption (3) and the continuity assumption (4). If the step sequence $(\gamma_n)_{n\geq 1}$ satisfies

$$\sum_{n\geq 1} \gamma_n = +\infty, \quad \gamma_n = o(n^{\underline{\alpha}-1}) \quad \text{and} \quad \sum_{n\geq 1} n^{1-\underline{\alpha}} \max(\gamma_n^2, |\gamma_n - \gamma_{n+1}|) < \infty$$

where $\underline{\alpha} := \min_{i\in I_N} \alpha_i \in (0,1)$, then the algorithm defined by (10) with growth control parameter $\vartheta \in (0, 2/3)$ *a.s.* converges towards $r^\infty = \mathrm{argmax}_{\mathcal{P}_N} \Phi$.

Example

An example of process that satisfies the assumptions of the Theorem 3 is the exponential of Ornstein-Uhlenbeck process (in continuous time) or an auto-regressive process (in discrete time). Let $(Y^n)_n$ be a sequence defined by

$$\forall n, \quad Y^n = (V, D_1^n, \ldots, D_N^n) = e^{X^n}$$

where $X^n = (X_0^n, \ldots, X_N^n)$ satisfies the recursive equation of $AR(1)$,

$$X^{n+1} = m + AX^n + \Sigma\epsilon_{n+1}$$

with $\|A\| < 1$, $\epsilon_n \sim \mathcal{N}(0, Id_{N+1})$ and $\mathrm{rk}(\Sigma) = N + 1$.
$(Y^n)_n$ is geometrically α-mixing at rate $\|A\|^n$ hence ergodic at rate $\frac{1}{2} - \epsilon$.

Outline

Description

This procedure originally introduced by R. Berenstein and C.-A. Lehalle is based on a

$$reinforcement mechanism.$$

Let I_i^n be the cost induced by the execution of the order sent to dark pool i at time n.

- The proportion r_i^n of the global order V^{n+1} to be sent to the dark pool i for execution at time $n+1$ is defined as proportional to this profit $i.e.$ by

$$\forall i \in I_N, \quad r_i^n := \frac{I_i^n}{\sum_j I_j^n}.$$

- The updating of the random vector I^n is as follows

$$\forall n \geq 0, \forall i \in I_N, \quad I_i^{n+1} = I_i^n + \rho_i \min\left(r_i^n V^{n+1}, D_i^{n+1}\right), \; I_i^0 = 0.$$

A new formulation

Elementary computations show that the algorithm can be written directly in a recursive way in terms of the vector-valued-state-variable

$$X^n = \frac{I^n}{n}$$

$$X_i^{n+1} = X_i^n - \frac{1}{n+1}\left(X_i^n - \rho_i \min\left(\frac{X_i^n}{\bar{X}^n} V^{n+1}, D_i^{n+1}\right)\right), \; i \in I_N,$$

where

$$\bar{X}^n = \sum_{j=1}^{N} X_j^n = \frac{1}{n}\sum_{j=1}^{n} I_j^n$$

and

$$r_i^n = \frac{X_i^n}{\bar{X}^n}.$$

This is a standard form for a stochastic algorithm (with step $\gamma_n = \frac{1}{n}$).

Existence of an equilibrium

Proposition

Let $N \geq 1$. Assume that (3) holds for every couple (V, D_i), $i \in I_N$.

(a) There exists a $x^* \in \mathbb{R}_+^N$ such that

$$\bar{x}^* := \sum_{i \in I_N} x_i^* > 0 \quad \text{and} \quad \varphi_i\left(\frac{x_i^*}{\bar{x}^*}\right) = x_i^*, \quad i \in I_N. \tag{14}$$

(b) Let $\psi_i := \frac{\varphi_i(u)}{u}$, $\quad u > 0, \quad i \in I_N, \quad \psi(0) = \varphi'(0) = \rho \mathbb{E} V \mathbf{1}_{\{D>0\}}$.
Assume that for every $i \in I_N$, ψ_i is (continuous) decreasing on $[0, \infty)$ and

$$(\mathcal{C}') \equiv \sum_{i \in I_N} \psi_i^{-1}(\min_{i \in I_N} \varphi_i'(0)) < 1. \tag{15}$$

Then there exists $x^* \in int(\mathcal{P}_N)$ satisfy (14).

Corollary 1

If the functions ψ_i are continuous and decreasing and the rebate coefficients ρ_i are equal (to 1) and if $\mathbb{P}(D_i = 0) = 0$ for every $i \in I_N$, then there exists an equilibrium point lying inside $int(\mathcal{P}_N)$.

Proposition 3

An equilibrium x^* satisfying (14) is locally uniformly attracting as soon as

$$\sum_{j \in I_N} \frac{x_j^*}{(\bar{x}^*)^2} \varphi_j'\left(\frac{x_j^*}{\bar{x}^*}\right) < 1 - \frac{1}{\bar{x}^*} \max_{i \in I_N} \varphi_i'\left(\frac{x_i^*}{\bar{x}^*}\right)$$

where $\bar{x}^* = \sum_{i \in I_N} x_i^*$.

A competitive system

A *competitive differential system* $\dot{x} = h(x)$ is a system in which the field $h : \mathbb{R}^N \mapsto \mathbb{R}^N$ is differentiable and satisfies

$$\forall x \in \mathbb{R}^N,\ \forall\, i, j \in I_N,\ i \neq j, \qquad \frac{\partial h_i}{\partial x_j}(x) > 0.$$

As concerns the reinforcement algorithm, the mean function h is given by

$$h : x \longmapsto \left(x_i - \varphi_i \left(\frac{x_i}{\sum_j x_j} \right) \right)_{1 \leq i \leq N}, \tag{16}$$

and under the standard differentiability assumption on the functions φ_i's,

$$\forall x \in \mathbb{R}^N, \quad \frac{\partial h_i}{\partial x_j}(x) = \varphi_i' \left(\frac{x_i}{x_1 + \cdots + x_N} \right) \frac{x_i}{(x_1 + \cdots + x_N)^2} > 0.$$

Advantages and Drawbacks

- Drawbacks
  - No hope to prove that all the equilibrium points lie in the interior of $\mathcal{P}_N$ since one may always adopt an execution strategy which boycotts a given dark pool or, more generally N_0 dark pools. Elementary combinatorial arguments show that there are *at least* $2^N - 1$ equilibrium points.
  - As a competitive system, the algorithm has possibly a non converging behaviour even in presence of a single (attracting) equilibrium. This is to be compared to their *cooperative* counterparts (with negative non diagonal partial derivatives).
- Advantages
  - on the contrary of the optimization algorithm, the reinforcement algorithm naturally lives in the simplex $\mathcal{P}_N$.

Outline

Abundance and Shortage

We compare the behaviour of both algorithms in different settings : the *IID* one, the *ERG* one and with pseudo-real data (whose construction is explained below).

We examine specifically two situations : abundance and shortage.

- What we call "abundance" is the fact that the mean of V is less that the sum of the means of the D_i, *i.e.*
$$\mathbb{E}V \leq \sum_{i=1}^{N} \mathbb{E}D_i$$
,
- and the "shortage" is the situation where we have the contrary, *i.e.*
$$\mathbb{E}V > \sum_{i=1}^{N} \mathbb{E}D_i.$$

The most interesting setting to compare them is the shortage since it is the situation the most common in the market.

Comparison Criterions

We present the performances of both algorithms and compare them to the strategy devised by an insider "oracle" who would know the true values of V and D_i. This "oracle" strategy is the best possible allocation.

$$\forall n \geq 1, \quad \min\left(V^n, \sum_{i=1}^{N} D_i^n\right).$$

So we introduce in the following figures

- the allocation coefficients of the optimization algorithm and the reinforcement algorithm (just drawn in the *IID* setting as example because the best way to compare the two algorithms is to look at their performances),

Comparison Criterions

- the ratios between the executed quantity and the sent quantity for the three algorithms (we name it *satisfaction*), *i.e.* for every $n \geq 1$,
  - $\dfrac{\sum_{i=1}^{i_0-1} \rho_i D_i + \rho_{i_0}\left(V - \sum_{i=1}^{i_0-1} \rho_i D_i\right)}{V^n}$ for the oracle, where $\rho_1 < \rho_2 < \cdots < \rho_N$, and i_0 such that $\sum_{i=1}^{i_0-1} D_i < V \leq \sum_{i=1}^{i_0} D_i$.
  - $\dfrac{\sum_{i=1}^{N} \rho_i \min\left(r_i^n V^n, D_i^n\right)}{V^n}$ for both algorithms.

- the ratios between the satisfaction index of both optimization and reinforcement algorithms and that of the oracle, *i.e.* for every $n \geq 1$

$$\frac{\sum_{i=1}^{N} \rho_i \min\left(r_i^n V^n, D_i^n\right)}{\sum_{i=1}^{i_0-1} \rho_i D_i + \rho_{i_0}\left(V - \sum_{i=1}^{i_0-1} \rho_i D_i\right)}.$$

With log-normal simulated data (Shortage)

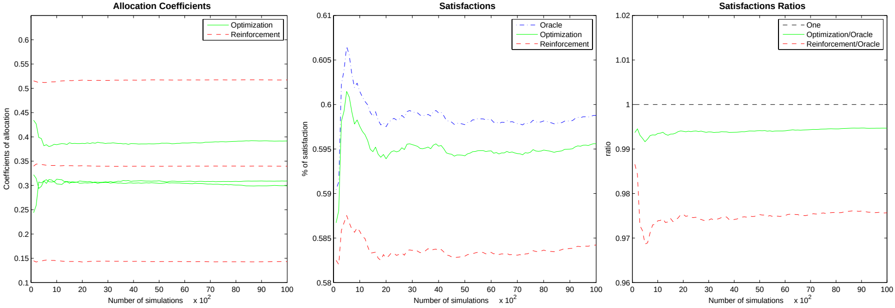


Figure: Case $N = 3$, $m_V = \frac{3}{2}\sum_{i=1}^{N} m_{D_i}$, $m_{D_i} = i$, $\sigma_V = 1$, $\sigma_{D_i} = 1$, $1 \leq i \leq N$.

Satisfaction evolution according to variance variation

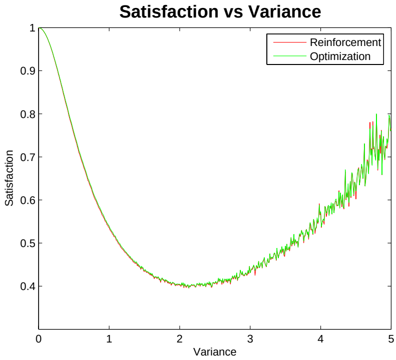


Figure: The satisfaction decreases as the variance increases between 0 and 2.5, then increases but less smoothy and is perturbated.

Exponential of OU

The quantity V and D_i, $i \in I_N$, are exponentials of an Ornstein-Uhlenbeck process, *i.e.*

$$X^{n+1} = m + AX^n + B\Xi^{n+1},$$

where $\|A\| < 1$, $BB^* \in Gl(d, \mathbb{R})$ and

$$m = \begin{pmatrix} m_1 \\ \vdots \\ m_{N+1} \end{pmatrix} \in \mathbb{R}^{N+1}, \quad \Xi^{n+1} = \begin{pmatrix} \Xi_1^{n+1} \\ \vdots \\ \Xi_{N+1}^{n+1} \end{pmatrix} \sim \mathcal{N}\left(0, I_{N+1}\right) \ i.i.d.,$$

$$e^{X^n} = \begin{pmatrix} V^n \\ D_1^n \\ \vdots \\ D_N^n \end{pmatrix}.$$

Numerical Data

The initial value of the algorithms is $r_i^0 = \frac{1}{N}$, $1 \le i \le N$ and we set

$$\rho = \begin{pmatrix} 0.95 \\ 0.97 \\ 0.99 \end{pmatrix}, \quad m = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0.7 & 0.01 & 0.01 & 0.01 \\ 0.01 & 0.3 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.2 & 0.01 \\ 0.01 & 0.01 & 0.01 & 0.1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.02 & 0 & 0 & 0 \\ 0.01 & 0.9 & 0 & 0 \\ 0.01 & 0.01 & 0.6 & 0 \\ 0.01 & 0.01 & 0.01 & 0.3 \end{pmatrix}.$$

Numerical Results (Shortage)

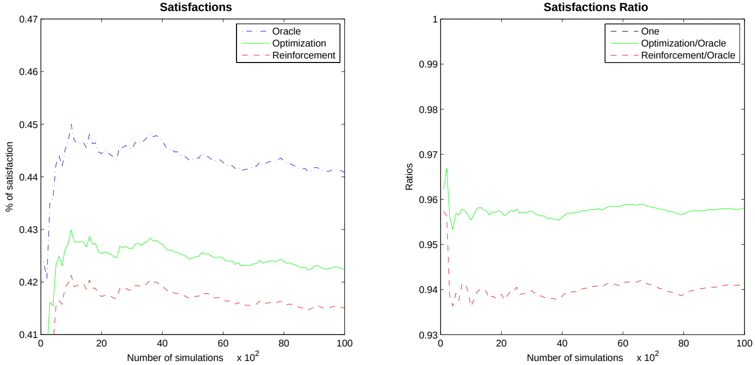


Figure: Case $N = 3$, $m_V \geq \sum_{i=1}^{N} m_{D_i}$, $\sigma_V \geq 1$, $\sigma_{D_i} \geq 1$, $1 \leq i \leq N$.

Generation of pseudo-real data

We consider an asset volume V which we want buy or sell, and the volumes which are offered by the N dark pools $(D_i)_{1\le i\le N}$. We have picked up data on the market for V and the D_i are building from the N assets which are the most correlated with V, denoted by S_i, $i=1,\dots,N$, and V by the mixing function

$$\forall 1 \le i \le N, D_i := \beta_i \left((1-\alpha_i)V + \alpha_i S_i \frac{\mathbb{E}V}{\mathbb{E}S_i} \right)$$

where

- α_i, $i=1,\dots,N$ are the mixing coefficients,
- β_i, $i=1,\dots,N$ some weights.

So

$$\mathbb{E}(D_i) = \beta_i \mathbb{E}(V).$$

Abundance and Shortage Cases

- If $\sum_{i=1}^{N} \beta_i < 1$, then $\mathbb{E}\left[\sum_{i=1}^{N} D_i\right] < \mathbb{E}V$: this is a shortage situation and we use the algorithm to find the optimal allocation.
- The simulation presented here have been made with the asset BNP and the four most correlated assets with BNP, so $N = 4$.

The data used extend on 11 days. To explain the changes in the response of the algorithms, we have underlined the days by drawing vertical lines to separate the days of execution. We place in the shortage situation : we set

$$\rho = \begin{pmatrix} 0.94 \\ 0.96 \\ 0.98 \\ 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.2 \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} 0.4 \\ 0.6 \\ 0.8 \\ 0.2 \end{pmatrix}.$$

Shortage case with pseudo-real data

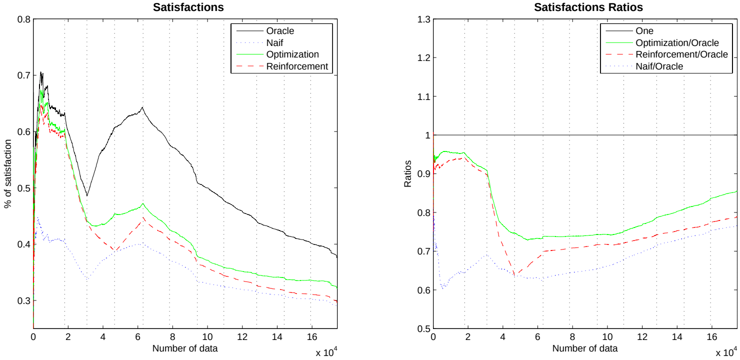


Figure: Case $N = 4$, $\sum_{i=1}^{N} \beta_i < 1$, $0 < \alpha_i \leq 0.2$ and $r_i^0 = 1/N$ $1 \leq i \leq N$

Outline

Toward more realistic mean execution functions

One natural idea is to take into account that the rebate may depend on the quantity rV sent to be executed by the dark pool. The mean execution function of the dark pool can be modeled by

$$\forall\, r \in [0,1], \qquad \varphi(r) = \mathbb{E}(\rho(rV)\min(rV, D)) \tag{17}$$

where the rebate function ρ is a non-negative, bounded, non-decreasing right differentiable function.

For the sake of simplicity, we assume that (V, D) satisfies (4). The right derivative of φ reads

$$\varphi'_r(r) = \mathbb{E}\left(\rho'_r(rV)V\min(rV, D)\right) + \mathbb{E}\left(\rho(rV)V\mathbf{1}_{\{rV<D\}}\right), \tag{18}$$

with in particular $\varphi'(0) = \rho(0)\,\mathbb{E}(V\mathbf{1}_{\{D>0\}}) > 0$ as above.

The main gap is to specify the function ρ so that φ remains concave. But the choice for ρ turns out to strongly depend on the (unknown) distribution of the random variable D.

Example: $V, D \sim \mathcal{E}(\lambda)$ independent. The function g is defined by

$$g(u) := \mathbb{E}(u \wedge D) = \frac{1 - e^{-u\lambda}}{\lambda},\ u \geq 0$$

so that, the execution function

$$\varphi(r) = \mathbb{E}(\rho(rV)g(rV))$$

will be concave as soon as the function $\rho\, g$ is so. Typical choices are $\rho = g^{\theta}$ with $\theta \in (0, \lambda]$ which may appear as not very realistic since the rebate function is a structural feature of the different dark pools.

A responsive dark pool

The dark pool may take into account the volume rV to decide which quantity will really executed rather than simply the *a priori* deliverable quantity D. One reason for such a behaviour is that the dark pool may wish to preserve the possibility of future transactions with other clients. So we introduce a *delivery function* $\psi : \mathbb{R}_+^2 \to \mathbb{R}_+$, non-decreasing and concave w.r.t. its first variable and satisfying $0 \le \psi(x, y) \le y$, so that the new mean execution function is as follows:

$$\varphi(r) = \rho\,\mathbb{E}\left(\min(rV, \psi(rV, D))\right). \tag{19}$$

It is clear that the function φ is concave (as the minimum of two concave functions) and bounded. In this case, the first (right) derivative of φ reads

$$\varphi'_r(r) = \rho\,\mathbb{E}\left(V\left(\mathbf{1}_{\{rV<\psi(rV,D)\}} + \psi'_x(rV, D)\mathbf{1}_{\{rV\ge\psi(rV,D)\}}\right)\right) \tag{20}$$

where ψ'_x denotes the right derivative with respect to x. In particular $\varphi'_r(0) = \rho\,\mathbb{E}(V\mathbf{1}_{\{D>0\}}) > 0$.

Example

We consider for modelling the quantity delivered by the dark pool i a function where we can define a minimal quantity required to begin to consum D_i, namely

$$\psi_i(rV, D_i) = D_i \mathbf{1}_{\{rV > s_i D_i\}}$$

where s_i is a parameter of the dark pool i assumed to be deterministic.

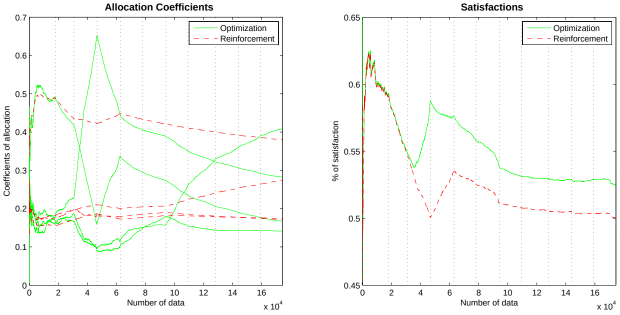


Figure: Pseudo-real data with $N = 4$, $\sum_{i=1}^{N} \beta_i < 1$, $0 < \alpha_i \leq 0.2$ and $r_i^0 = 1/N$, $1 \leq i \leq N$, $s = (0.3, 0.2, 0.2, 0.3)^t$.

Optimization *vs* reinforcement ?

For practical implementation what conclusions can be drawn form our investigations on both procedures.

- Both reach quickly a stabilization/convergence phase close to optimality.
- The reinforcement algorithm leaves the simplex structurally stable which means the proposed dispatching at each time step is realistic whereas the stochastic Lagrangian algorithm may sometimes need to be corrected.
- However, in a high volatility context, the stochastic Lagrangian algorithm clearly prevails with performances that may be significantly better performance.
- This optimization procedure also relies on established convergence results in a rather general framework (stationary α-mixing input data).
- Given the computational cost of these procedures which is close to zero, a good strategy is probably to implement both in parallel.

Variants of the implementation

- Resetting the step
- Constant step *vs* decreasing step
- Convergence *vs* pursuit

References

- ALMGREN, R. F. AND CHRISS N. (2000). Optimal execution of portfolio transactions, *Journal of Risk*, 3(2):5-39.
- ALMGREN R. AND HARTS B. (2008). A dynamic algorithm for smart order routing, Technical report, StreamBase.
- BENVENISTE M., MÉTIVIER M., PRIOURET P. (1987). *Adaptive Algorithms and Stochastic Approximations*, Springer Verlag, Berlin, 365p.
- BERENSTEIN R. (2008). *Algorithmes stochastiques, microstructure et exécution d'ordres*, Master 2 internship report (Quantitative Research, Dir. C. Lehalle, CA Cheuvreux), *Probabilités & Finance*, UPMC-École Polytechnique, 2008.
- DOUKHAN P. (1991). *Mixing: Properties and Examples*, Lecture Notes Statist. **85**, Springer, New York, 142p.
- DUFLO M. (1997). *Random Iterative Models*. Series: Stochastic Modelling and Applied Probability, Vol. 34. XVI, 385 p., Springer,

- Foucault, T. and Menkveld, A. J. (2006). Competition for order flow and smart order routing systems, *Journal of Finance*, **63**(1):119-158.
- Hirsch M., Smith H. (2004). *Monotone Dynamical systems*, monography, 136p.
- Kushner H.J., Yin G.G. (1997). Stochastic approximation and recursive algorithms and applications, New York, Springer, 496p; and 2^{nd} edition, 2003.
- Laruelle S., Pagès G. (2009). Stochastic Approximation with Ergodic Innovation Revisited, pré-pub LPMA.
- Lehalle C.-A. (2009). Rigorous strategic trading: Balanced portfolio and mean-reversion, *The Journal of Trading*, **4**(3):40-46.
- Pagès G. (2001). Sur quelques algorithmes récursifs pour les probabilités numériques, *ESAIM P&S*, **5**:141-170.